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CLASSES OF GRAPHS THAT EXCLUDE A TREE AND A CLIQUE AND ARE NOT VERTEX RAMSEY

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A class Γ of graphs is vertex Ramsey if for all $H \in \Gamma$ there exists $G \in \Gamma$ such that for all partitions of the vertices of G into two parts, one of the parts contains an induced copy of H. Forb(T,K) is the class of graphs that induce neither T nor K. Let $T(k,r)$ be the tree with radius r such that each nonleaf is adjacent to k vertices farther from the root than itself. Gyárfás conjectured that for all trees T and cliques K, there exists an integer b such that for all G in Forb(T,K), the chromatic number of G is at most b. Gyárfás' conjecture implies a weaker conjecture of Sauer that for all trees T and cliques K, Forb(T,K) is not vertex Ramsey. We use techniques developed for attacking Gyárfás' conjecture to prove that for all q, r and sufficiently large k, Forb$(T(k,r),K_q)$ is not vertex Ramsey.

0. Introduction

For a graph G let $\omega(G)$ be the number of vertices of the largest clique in G. For a graph or hypergraph G, let $\chi(G)$ be the chromatic number of G, i.e., the least number c such that the vertices of G can be colored with c colors so that there are no monochromatic edges. For a graph $G=(V,E)$ and $X \subset V$, let $G[X]$ denote the subgraph of G induced by X in G. Let Forb(T,K) be the class of graphs that induce neither T nor K. Let K_q be the clique on q vertices. Let $T(k,r)$ be the rooted tree with radius r such that each nonleaf has k sons. See Figure 1. The starting point for the research reported in this article is the following conjecture, due independently to Gyárfás [1] and Sumner [16].

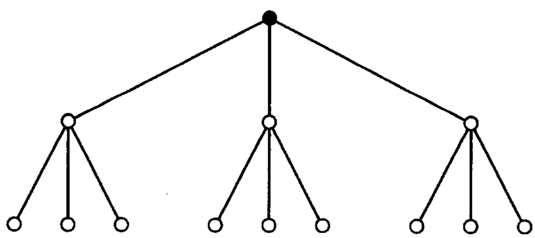

Figure 1. $T(3,2)$

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Conjecture 0.1. *For all trees T and cliques K, there exists an integer b such that for all G in* $\mathrm{Forb}(T,K)$, $\chi(G) \le b$.

While the general conjecture appears to be very difficult and has survived for twenty years, there have been some interesting partial results. The fundamental result in the subject is the following theorem due to Gyárfás, Szemerédi, and Tuza [3].

Theorem 0.2. *For every radius two tree T, there exists an integer b such that for all* $G \in \mathrm{Forb}(T,K_3)$, $\chi(G) \le b$.

To prove this theorem the authors used a technique that has developed into what we call the template technique. In a series of papers [6], [4], [8], [5] and [9], this technique has been modified in one way or another to prove similar results. The highlight, for the purposes of this article is the following extension of Theorem 0.2, due to Kierstead and Penrice [6].

Theorem 0.3. *For every radius two tree T and clique K, there exists an integer b such that for all* $G \in \mathrm{Forb}(T,K)$, $\chi(G) \le b$.

One important result in this area that does not use the template technique is the following theorem due to Scott [15].

Theorem 0.4. *For every tree T and clique K, if a graph G has sufficiently large chromatic number, then G induces either K or a subdivision of T.*

Independently Rödl, Sauer, and X. Zhu [10], [11], [12], [13], [14] studied classes of graphs to determine whether or not they were vertex Ramsey. A class Γ of graphs is vertex Ramsey if for every graph $H \in \Gamma$, there exists a graph $G \in \Gamma$ such that whenever the vertices of G are partitioned into two parts, one of the parts induces a copy of H. This property can be phrased in terms of hypergraph coloring. For any graphs $G=(V,E)$ and H, let $\binom{G}{H}$ be the hypergraph on the same vertex set V as G such that $X \subset V$ is an edge of $\binom{G}{H}$ iff $G[X]$ is isomorphic to H. Let $\chi_H(G)$ denote the chromatic number of $\binom{G}{H}$. It is routine to show that:

Proposition 0.5. *A class of graphs Γ is not vertex Ramsey if and only if there exists $H \in \Gamma$ and an integer b such that for all $G \in G$, $\chi_H(G) < b$.*

The combined work of Nešetřil, Rödl, Sauer, and X. Zhu determined whether or not $\mathrm{Forb}(A)$ is vertex Ramsey for nearly all graphs A. For any pair of graphs A and B such that A is disconnected and the complement of B is disconnected, Sauer [12] determined that $\mathrm{Forb}(A,B)$ is not vertex Ramsey, except for three special pairs for which it is vertex Ramsey. The simplest unknown case would seem to be when A is almost disconnected and the complement of B is totally disconnected. This led Sauer [12] to conjecture that:

Conjecture 0.6. *For all trees T and cliques K,* $\mathrm{Forb}(T,K)$ *is not vertex Ramsey.*

It is easily seen, using Proposition 0.5 that Conjecture 0.1 implies Conjecture 0.6. In this article we shall use our latest version of the template technique to prove that for all positive integers q and r and sufficiently large integers k, $\mathrm{Forb}(T(k,r),K_q)$ is not vertex Ramsey. In Section 1 we give an abstract definition of what it means for a graph H to be a (k,s)-template for a graph G. It is immediate from this definition that for all $G\in\mathrm{Forb}(T(k,2),K_3)$, and sufficiently large t, $K_{t,t}$ is a $T(k,1)$-template for G. Then we show that if G, $H\in\mathrm{Forb}(T(k,r))$ and H is a $(k,r-1)$-template for G, then $\chi_H(G)$ is bounded above by a function of T, $\omega(G)$, and H. This is all an abstraction of the main argument in [3], where templates were never defined, but their role was played by large $K_{t,t}$. In [3] the proof of Theorem 0.2 is completed by the observation of Rödl [9] that for all $G\in\mathrm{Forb}(T(k,2),K_3)$, if $\chi_{K_{t,t}}(G)$ is bounded above, then $\chi(G)$ is bounded above. Thus Section 1 specializes to the main argument in [3]. If we could prove that for all positive integers q and r and sufficiently large integers k, there exists a graph $H\in\mathrm{Forb}(T(k,r),K_q)$ such that for every $G\in\mathrm{Forb}(T(k,r),K_q)$ with sufficiently large chromatic number, H is a $(k,r-1)$-template for G, then we could prove that Conjecture 0.1 is true. We do not know how to do this. However, in Section 2 we do show that for all positive integers q and r, and sufficiently large k, there exists $H\in\mathrm{Forb}(T(k,r),K_q)$ such that for every $G\in\mathrm{Forb}(T(k,r),K_q)$, if H is an induced subgraph of G, then H is a $T(k,r-1)$-template for G. It then follows from the work in Section 1 that Conjecture 0.6 is true in the case that $T=T(k,r)$ and $K=K_q$.

We shall use the following additional notation. For a positive integer n, let $[n]$ denote the set $\{1,\dots,n\}$. Let $R(k,q)$ denote the least integer n such that every graph on n vertices has an independent set of size k or a clique of size q. For a graph $G=(V,E)$ and a subset $W\subset V$, let $N(W)$ denote $N(W)=\{v\in V-W : v$ is adjacent to w, for some $w\in W\}$. When $W=\{v\}$, we write $N(v)$ instead of $N(\{v\})$. We use the notation $G\approx H$ to indicate that G is isomorphic to H.

1. Definition of Templates

Fix positive integers k, r, and q. Set $T=T(k,r)$, $T'=T(k,r-1)$, and $\Gamma=\mathrm{Forb}(T,K_q)$. Let $t=(k^{r+1}-1)/(k-1)$, the number of vertices of T, and $t'=(k^r-1)/(k-1)$, the number of vertices of T'. Let $G=(V,E)$ be a graph. For a vertex $v\in V$ and a subset $W\subset V$, v is *adjacent* to W if v is adjacent to some vertex in W. Similarly, v is *strongly adjacent* to W if v is adjacent to at least $\alpha=2qk$ vertices in W and v is *very strongly adjacent* to W if v is adjacent to at least $\beta=kt'\alpha$ vertices of W. Let $N^s(W)=\{v\in V-W : v$ is strongly adjacent to $W\}$.

Definition 1.0. Let $G=(V,E)$ be a graph. Another graph $H=(W,F)$ is a $(k,r-1)$-*template* for G if H is an induced subgraph of G and the following three conditions hold:

(1) For all $v\in N(W)$, for all $X\subset W$, with $|X|\le\gamma=k\beta$, if $(W\cap N(v))-X\neq\emptyset$, then there exist $w\in(W\cap N(v))-X$ and $S\subset W-(N(v)\cup X)$ such that $G[S\cup\{w\}]\approx T'$ with root w.

(2) For all $v \in N(W)$, there exists $w \in (W \cap N(v))$ such that for all $X \subset W - \{w\}$, with $|X| \leq \gamma$, there exists $S \subset W - (N(v) \cup X)$ such that $G[S \cup \{w\}] \approx T'$ with root w.

(3) For all $v \in N^s(W)$, there exists $S \subset W$ such that $G[S \cup \{v\}] \approx T'$ with root v.

When k and r are fixed, a template will mean a $(k, r-1)$-template.

Suppose that G is a triangle-free graph and $H \approx K_{t,t}$ is an induced subgraph of G, with $t = k + \gamma$. Then it is easily seen that H is a $(k,1)$-template for G. The definition of template and the following theorem are an abstraction of the main argument in [6], where $K_{t,t}$ plays the role of the template.

Theorem. 1.1. *Suppose that $G = (V,E) \in \Gamma$ and H is a graph such that for all $X \subset V$, if $G[X] \approx H$, then $G[X]$ is a $(k, r-1)$-template for $G \in \Gamma$ and for all $v \in V$, $\chi_H(N(v)) \leq \lambda = \lambda_H(G)$. Then $\chi_H(G) \leq 1 + hd + h\lambda 2t'nR(k,q)$, where $h = |W|$, $d = 2kR(k,q)$, and $n = d - 1 + 2t'dR(k,q)$.*

Proof. Define M_1, N_1, V_1, ..., M_s, N_s, V_s, L as follows. Set $M_0 = \emptyset = N_0$ and $V_1 = V$. Suppose we have defined M_j, N_j, V_{j+1}, for all $j < i$. If $G[V_i]$ does not induce H, then set $s = i-1$ and $L = V_i$; otherwise choose $M_i \subset V_i$ such that $G[M_i] \approx H$ and set $N_i = N^s(M_i) \cap V_i$ and $V_{i+1} = V_i - (N_i \cup M_i)$. Then $\{M_1, N_1, \ldots, M_s, N_s, L\}$ is a partition of V. Note that for $j < i$, no vertex in V_i is strongly adjacent to M_j. The *(very strong) template degree* of a vertex $v \in V$ is $|\{i : v$ is (very strongly) adjacent to $M_i\}|$. The *neighborhood degree* of a vertex $v \in V$ is $|\{i : v$ is adjacent to $N_i\}|$. We shall need the following three lemmas.

Lemma 1.2. *For every vertex $v \in V$, the very strong template degree of v is less than k.*

Proof. Suppose that $v \in V$ and $I = \{i_1 < \ldots < i_k\}$ is a k-subset of $[s]$ such that for every $i \in I$, v is very strongly adjacent to M_i. We shall obtain a contradiction to $G \in \Gamma$ by showing that v is the root of a copy of T. To do this, for each $i \in I$, we find $v_i \in M_i$ and $S_i \subset M_i - \{v_i\}$ such that for all distinct i, $j \in I$:

(i) v is adjacent to v_i;

(ii) $G[S_i \cup \{v_i\}] \approx T'$ with root v_i;

(iii) v is not adjacent to S_i; and

(iv) $S_i \cup \{v_i\}$ is not adjacent to $S_j \cup \{v_j\}$.

We construct the v_i and S_i by reverse induction. Let $i \in I$ and suppose that we have constructed v_j and S_j for all $j \in I$ with $j > i$. Let $X = \{x \in M_i : x$ is adjacent to $S_j \cup \{v_j\}$, for some $j \in I$ with $j > i\}$. Say $i = i_a$. Then $|X| \leq (k-a)t'(\alpha - 1) < kt'\alpha = \beta$, since for each of the $k - a$ indices $j > i$, each of the t' vertices of $S_j \cup \{v_j\}$ is adjacent to less than α elements of M_i. Since v is very strongly adjacent to M_i, $(M_i \cap N(v)) - X \neq \emptyset$. By (1) we can find $v_i \in (M_i \cap N(v)) - X$ and $S_i \subset M_i - (N(v) \cup X)$ such that $G[S_i \cup \{v_i\}] \approx T'$ with root v_i. Since $(S_i \cup \{v_i\}) \cap X = \emptyset$, $S_i \cup \{v_i\}$ is not adjacent to $S_j \cup \{v_j\}$, for all $j \in I$ with $j > i$. This completes the construction of the S_i and the proof. ∎

Lemma 1.3. *For every vertex $v \in V$, the template degree of v is less than $d = 2kR(k,q)$.*

Proof. Suppose that $v \in V$ and $I = \{i_1 < \ldots < i_d\}$ is a d-subset of $[s]$ such that for every $i \in I$, v is adjacent to M_i. We shall obtain a contradiction to $G \in \Gamma$ by showing that v is the root of a copy of T. To do this we find a k-subset $J \subset I$ and for every $i \in J$, we find $v_i \in M_i$ and $S_i \subset M_i$ such that for all distinct $i,\ j \in J$:

(i) v is adjacent to v_i;

(ii) $G[S_i \cup \{v_i\}] \approx T'$ with root v_i;

(iii) v is not adjacent to S_i; and

(iv) $S_i \cup \{v_i\}$ is not adjacent to $S_j \cup \{v_j\}$.

By (2), for each $i \in I$, there exists $v_i \in (M_i \cap N(v))$ such that for all $X \subset M_i - \{v_i\}$, with $|X| \leq \gamma$, there exists $S \subset M_i - (N(v) \cup X)$ such that $G[S \cup \{v_i\}] = T'$ with root v_i. By Lemma 1.2, each v_i has very strong template degree less than k. Thus there exists an $R(k,q)$-subset $J' \subset I$ such that for all $i,\ j \in J$, v_i is not very strongly adjacent to M_j. (Define a digraph $D = (I,A)$ by $(i,j) \in A$ iff v_i is very strongly adjacent to M_j. Since the out degree of D is less than k, D has an independent set of size $d/(2k-1)$.) Since G has clique size less than q, there exists a k-subset $J \subset J'$ such that $\{v_i : i \in J\}$ is independent. Next we construct the S_i by reverse induction. Let $i \in J$ and suppose that we have constructed S_j for all $j \in J$ with $j > i$ so that for all $i' \in J$ with $i' \leq i$, $v_{i'}$ is not adjacent to $S_j \cup \{v_j\}$. Let

$$X = \{x \in M_i : x \text{ is adjacent to } S_j \cup \{v_j\},\ \text{for some } j \in J \text{ with } j > i\} \cup$$
$$\{x \in M_i : x \text{ is adjacent to } v_{i'},\ \text{for some } i' \in J,\ \text{with } i' < i\}$$

Say $i = i_a$. Then $|X| < (k-a)t'(\alpha - 1) + (a-1)\beta < k\beta = \gamma$, since for each of the $k-a$ indices $j > i$, each of the t' vertices of $S_j \cup \{v_j\}$ is adjacent to less than α vertices of M_i and for each of the $a-1$ vertices $v_{i'}$, with $i' < i$, $v_{i'}$ is adjacent to less than β vertices of M_i. By the choice of the v_i and using (2), we can find $S_i \subset M_i - (N(v) \cup X)$ such that $G[S_i \cup \{v_i\}] \approx T'$ with root v_i. Since $S_i \cap X = \emptyset$, $S_i \cup \{v_i\}$ is not adjacent to $S_j \cup \{v_j\}$, for all $j \in I$ with $j > i$ and $v_{i'}$ is not adjacent to S_i, for all $j \in J - \{i\}$. This completes the construction of the S_i and the proof. ∎

Lemma 1.4. *For every vertex $v \in V$, the neighborhood degree of v is less than $n = d - 1 + 2t'dR(k,q)$.*

Proof. Suppose that $v \in V$ and $I = \{i_1 < \ldots < i_n\}$ is an n-subset of $[s]$ such that for every $i \in I$, v is adjacent to $v_i \in N_i$. We shall obtain a contradiction to $G \in \Gamma$ by showing that v is the root of a copy of T. To do this we shall find a k-subset $J \subset I$ and for every $i \in J$, we shall find $S_i \subset M_i$ such that for all distinct $i,\ j \in J$:

(ii) $G[S_i \cup \{v_i\}] \approx T'$ with root v_i;

(iii) v is not adjacent to S_i; and

(iv) $S_i \cup \{v_i\}$ is not adjacent to $S_j \cup \{v_j\}$.

By (3), for each $i\in I$, we can find $S_i\subset M_i$ such that $G[S_i\cup\{v_i\}]\approx T'$ with root v_i. By Lemma 1.3, every vertex has template degree less than d. Thus there exists a $2t'dR(k,q)$-subset $J'\subset I$ such that for all $i\in J'$, v is not adjacent to M_i. Also, there exists an $R(k,q)$-subset $J''\subset J'$ such that for all distinct i, $j\in J''$, $S_i\cup\{v_i\}$ is not adjacent to S_j. Finally, since $\omega(G)<q$, there exists a k-subset $J\subset J''$ such that $\{v_i:i\in J\}$ is independent. This completes the construction of the S_i and the proof. ∎

We are now ready to finish the proof of the theorem by exhibiting a proper coloring of $\binom{G}{H}$ using at most $1+hd+h\lambda 2t'nR(k,q)$ colors. First, partition V into $L\cup M\cup N$, where $M=\cup\{M_i:i\in[s]\}$ and $N=\cup\{N_i:i\in[s]\}$. Color all the vertices in L with color 0. Since $G[L]$ does not induce H, this is a proper coloring of $\binom{G[L]}{H}$. Let $M_i=\{v_{i,1},\dots,v_{i,h}\}$. Color each $v_{i,j}$, with color (a,j), where a is the least positive integer such that for all $i'<i$, if $v_{i'j}$ has color (a,j), then $v_{i,j}$ is not adjacent to $v_{i',j}$. By Lemma 1.3, $a\le d$. Since k, $r-1\ge 1$, H has an edge, and so this is a proper coloring of $\binom{G[M]}{H}$ with hd colors, all distinct from the color used on L. Thus it remains to show that we can properly color $\binom{G[N]}{H}$ with $h\lambda 2t'nR(k,q)$ new colors.

Let $N_{i,j}=\{v\in N_i:v$ is adjacent to $v_{i,j}$, but v is not adjacent to any $v_{i,j'}$, with $j'<j\}$. For each pair (i,j), let $f_{i,j}:N_{i,j}\to[\lambda]$ be a proper coloring of $\binom{G[N_{i,j}]}{H}$ and let $N_{i,j,b}=\{v\in N_{i,j}:f_{i,j}(v)=b\}$. Color each vertex in $N_{i,j,b}$ with color (a,j,b), where a is the least positive integer so that $G[\cup\{N_{i',j,b}:i'\le i\}]$ does not contain a monochromatic copy of H. Note that such an integer exists because $N_{i,j,b}$ does not contain a monochromatic copy of H.

To finish the proof, we must show that $a\le 2t'nR(k,q)$. Otherwise for all $c\in[2t'nR(k,q)]$, there exists a subset $Q_c\subset\cup\{N_{i',j,b}:i'\le i\}$ such that

(v) $G[Q_c]\approx H$,

(vi) $Q_c\cap N_{i,j,b}\neq\emptyset$, and

(vii) each vertex of $Q_c-N_{i,j,b}$ is colored (c,j,b).

We shall obtain a contradiction to $G\in\Gamma$ by showing that $v_{i,j}$ is the root of a copy of T. To do this, we shall find a k-subset $J\subset[2t'nR(k,q)]$ and for every $c\in J$, we shall find $v_c\in Q_c\cap N_{i,j,b}$ and $S_c\subset Q_c-N(v_{i,j})$ such that for all distinct c, $c'\in J$:

(i) $v_{i,j}$ is adjacent to v_c;

(ii) $G[S_c\cup\{v_c\}]\approx T'$ with root v_c;

(iii) $v_{i,j}$ is not adjacent to S_c; and

(iv) $S_c\cup\{v_c\}$ is not adjacent to $S_{c'}\cup\{v_{c'}\}$.

By (1), for each $c\in[2t'nR(k,q)]$, there exists $v_c\in Q_c\cap N(v_{i,j})$ and $S_c\subset Q_c-N(v_{i,j})\subset Q_c-N_{i,j,b}$ such that $G[S_c\cup\{v_c\})\approx T'$ with root v_c. By (vii), for all colors c and c', and indices $m \neq i$, if $Q_c \cap N_{m,j,b} \neq \emptyset$, then $Q_{c'} \cap N_{m,j,b} = \emptyset$. Thus by Lemma 1.4, each $S_c \cup \{v_c\}$ is adjacent to less than $t'n$ other $S_{c'}$. Thus there exists an $R(k,q)$-subset $J' \subset [2t'nR(k,q)]$ such that for all distinct $c, c' \in J'$, $S_c \cup \{v_c\}$ is not adjacent to $S_{c'}$. Since $\omega(G) < q$, there exists a k-subset $J \subset J'$ such that $\{v_c : c \in J\}$ is independent. This completes the proof. ∎

2. Existence of Templates

In this section we show that templates actually exist.

Theorem 2.1. *For all positive integers q and r, and sufficiently large integers k, there exists a graph $H \in \mathrm{Forb}(T(k,r), K_q)$ such that for all $G \in \mathrm{Forb}(T(k,r), K_q)$, if H is an induced subgraph of G, then H is a template for G.*

Proof. As in Section 1, let $T = T(k,r)$, $T' = T(k, r-1)$, and $\Gamma = \mathrm{Forb}(T(k,r), K_q)$. We first construct $H = H(q,r,k,n,p)$ in terms of fixed positive integer parameters q, r, k, and n, and a real parameter p, with $0 < p < 1$; then we show that for the proper choice of these parameters, H is a template for every graph in Γ that induces H. The construction of H is based on a simpler graph A. Let $A = A(q) = (W, F)$ be the graph defined by $W = [2q-1]$ and $F = \{ij : j = i \oplus m,\ m \in [q-2]\}$, where $\oplus$ denotes addition modulo $2q-1$. In other words, A is the $(q-2)$-power of a $(2q-1)$-cycle. We are now ready to define H. For each vertex $i \in W$ there will be a corresponding set $U(i)$ of vertices of H. The vertex set of H is $Y = \cup\{U(i) : i \in W\}$, where the $U(i)$ are pairwise disjoint sets of cardinality n. For all edges $ij \in F$, any vertex in $U(i)$ is adjacent to any vertex in $U(j)$ with probability p. All such decisions are made independently. There are no other edges in H. So far H is a random graph. We complete the construction of H by choosing a particular version of H that satisfies conditions (i), (ii), (iii), and (iv) of the following lemma. This is the only part of the argument where we use probability theory.

Lemma 2.2. *Suppose that for some integers b and c*

(1) $4q^2 n e^{-p^2 n/2} < 1/4$

(2) $4q^3 n^2 e^{-2p^4 n} < 1/4,$

(3) $4q^2 \binom{n}{b}^2 (1-p)^{b^2} < 1/4,$

(4) $2q \binom{n}{c}^{q-1} \left(1 - p^{q^2/2}\right)^{2c^2/q^2} < 1/4.$

Then the probability that H has the following four properties is greater than 0.

(i) *For all edges $ij \in F$ and vertices $y \in U(i)$, $|N(y) \cap U(j)| \geq pn/2$.*

(ii) *For all edges $ij, mj \in F$ (possibly $i = m$), and for all $y \in U(i)$ and $y' \in U(m) - \{y\}$, $|N(y) \cap N(y') \cap U(j)| \leq 2p^2 n$.*

(iii) *For all edges* $ij \in F$ *and all* b*-subsets* $X \subset U(i)$ *and* $X' \subset U(j)$*, there exists* $u \in X$ *and* $v \in X'$ *such that* u *is adjacent to* v *in* H*.*

(iv) *For all* c*-subsets* $X_1 \subset U(i\oplus 1), \ldots, X_{q-1} \subset U(i \oplus (q-1))$*, there exist* $x_1 \in X_1$*,* $\ldots$*,* $x_{q-1} \in X_{q-1}$*, such that* $(x_1, \ldots, x_{q-1})$ *is a clique in* H*.*

Proof. It suffices to show that the probability of any one of (i), (ii), (iii) or (vi) failing is less than 1/4.

(i) There are less than $4q^2n$ ways to choose an edge ij and a vertex $y \in U(i)$. For each of the n vertices $y' \in U(j)$, the events "yy' is an edge in H" are independent and have probability p. Let $X(j,y) = |N(y) \cap U(j)|$. Then $X(j,y)$ is the sum of n independent indicator random variables, each having probability p. So

$$\Pr(X(j,y) < pn - pn/2) < e^{-2(pn/2)^2/n}.$$

Thus the probability that (i) fails is less than $4q^2ne^{-p^2n/2} < 1/4$.

(ii) There are less than $4q^3$ ways to choose the edges ij and mj. For each of these choices, there are at most n^2 ways to choose y and y'. For each of the n vertices $z \in U(j)$, the events "both yz and $y'z$ are edges in H" are independent and have probability p^2. Let $X = X(j,y,y') = |N(y) \cap N(y') \cap U(j)|$. Then X is the sum of n independent indicator random variables, each having probability p^2. So

$$\Pr(X > p^2n + p^2n) < e^{-2(p^2n)^2/n}.$$

Thus the probability that (ii) fails is less than $4q^3n^2e^{-2p^4n} < 1/4$.

(iii) There are less than $4q^2$ ways to pick an edge ij. For each of these, there are $\binom{n}{b}^2$ ways to pick b-subsets $X \subset U(i)$ and $X' \subset U(j)$. For each of the b^2 pairs $(x,x') \in X \times X'$, the events "xx' is an edge in H" are independent and have probability p. Thus the probability that there is no edge between X and X' is less than $(1-p)^{b^2}$. Thus the probability that (iii) fails is at most $4q^2\binom{n}{b}^2(1-p)^{b^2} < 1/4$.

(iv) There are less than $2q$ ways to pick i. For each of these there are $\binom{n}{c}^{q-1}$ ways to choose the c-subsets $X_m \subset U(i \oplus m)$, $m \in [q-1]$. Say $X_m = \{(m,j) : j \in [c]\}$. For a function $s : [q-1] \to [c]$, let $P(s)$ be the event "$((1,s(1)), \ldots, (q-1, s(q-1)))$ is a clique in H". Then $\Pr(P(s)) = p^{\binom{q-1}{2}}$. Consider the graph $\mathbf{S}$ on $S = \{s : s : [q-1] \to [c]\}$, such that s is adjacent to t iff there exist distinct j, $m \in [q-1]$ such that $s(j) = t(j)$ and $s(m) = t(m)$. Clearly the degree of $\mathbf{S}$ is less than $c^{q-3}q^2/2$. Thus $\mathbf{S}$ has an independent subset T of size $|S|/(c^{q-3}q^2/2) = 2c^2/q^2$. The set of events $\{P(s) : s \in T\}$ is independent. Thus the probability that (iv) fails is at most

$$2q\binom{n}{c}^{q-1}\left(1 - p^{q^2/2}\right)^{2c^2/q^2} < 1/4. \qquad \blacksquare$$

We now set $n = k^{q^2 r}$ and $p = \frac{1}{24}k^{-r+2}$. Then (i) and (ii) hold for sufficiently large k. We also set $b = 2\log n/p$ and $c = q^3 \log n/p^{q^2/2}$. Then (iii) and (iv) hold. Thus we can, and do, choose H satisfying conditions (i), (ii), (iii), and (iv). Our next goal is to show that $H \in \mathrm{Forb}(T(k,r), K_q)$, for the proper choice of k. This follows from the next two lemmas.

Lemma 2.3. *Both $\omega(A) < q$ and $\omega(H) < q$.* ∎

Lemma 2.4. *If $b \leq k^{r-1}/(4q^2)$, then $H \in \mathrm{Forb}(T(k,r))$. In particular, for sufficiently large k, $H \in \mathrm{Forb}(T(k,r))$.*

Proof. Suppose to the contrary that H does induce T. Since T has k^{r-1} vertices at distance $r-1$ from its root, there exists a vertex $i \in W$ and a subset $Z \subset U(i)$ such that $|Z| = k^{r-1}/(2q)$ and each vertex in Z is at distance $r-1$ from the root of T. Each vertex in Z is adjacent to a leaf of T that is in $U(j)$, for some vertex j adjacent to i in A. Thus there exist a vertex $j \in W$ and sets $Z' \subset Z$ and $X' \subset U(j)$ such that $|Z'| = k^{r-1}/(4q^2) = |X'|$ and each vertex in Z' is adjacent to a leaf of T that is in X'. Let X be a subset of $Z - Z'$ of size $k^{r-1}/(4q^2)$. Then there are no edges between X and X' in T. Since T is an induced subgraph of H there are no edges between X and X' in H. But this contradicts (iii), since $b = k^{r-1}/(4q^2)$. Finally, note that for sufficiently large k, $b \leq k^{r-1}/(4q)$. ∎

Next we show that for sufficiently large k, H is a template for G, for all graphs $G \in \Gamma$ such that H is an induced subgraph of G. Fix such a graph $G = (V,E)$. Let $v \in V \cap N(H)$. Let $d = \max\{\gamma, c\}$. Call a vertex $y \in Y \cap N(v)$ *v-good*, if there exist vertices i, j, $m \in W$ (possibly $i = m$) such that $y \in U(i)$, ij and jm are edges in A, $|N(v) \cap U(j)| < d$, and $|N(v) \cap U(m)| < d$.

Lemma 2.5. *For any function $f: W \to [2]$, with $2 \in \mathrm{range}(f)$, either* (a) *there exists a path (i,j,m) in A such that $f(i) = 2$, and $f(j) = 1 = f(m)$ or* (b) *there exists a vertex $i \oplus 1$ such that for all $j \in [q-1]$, $f(i \oplus j) = 2$.*

Proof. Suppose not. Then there exists a vertex $i \in W$ such that $f(i) = 2$ and $f(i \oplus 1) = 1$. Without loss of generality, $f(2q-1) = 2$ and $f(1) = 1$. Then, since (a) fails, $(2q-1, 1, j)$ witnesses that $f(j) = 2$, for all $j \in \{2, \ldots, q-1\}$. Thus, since (b) fails, $f(q) = 1$. So, since (a) fails, $(q-1, q, j)$ witnesses that $f(j) = 2$, for all $j \in \{q+1, \ldots, 2q-2\}$. Since $f(2q-1) = 2$, (b) must hold. ∎

Lemma 2.6. *For every $v \in V \cap N(H)$, either* (c) *every vertex in $N(v) \cap Y$ is v-good or* (d) *there exists a vertex $i \in W$ such that there exist at least d vertices in $U(i)$ that are v-good.*

Proof. Let $f: W \to [2]$, where $f(i) = 1$ if $|N(v) \cap U(i)| < d$, and otherwise $f(i) = 2$. If for all vertices $i \in W$, $|N(v) \cap U(i)| < d$, then (c) holds. Otherwise by Lemma 2.5, either (a) or (b) holds. If (b) holds, then there exists a $i \in W$ such that

$|N(v) \cap U(i \oplus j)| \geq d$, for all $j \in [q-1]$. By (iv), for all $j \in [q-1]$, there exist vertices $y_{i\oplus j} \in N(v) \cap U(i \oplus j)$ such that $Q = (y_{i\oplus 1}, \ldots, y_{i\oplus q-1})$ is a clique in H. But, then $Q \cup \{v\}$ is a q-clique in G, contradicting $G \in \Gamma$. Thus (a) holds. Suppose (i,j,m) witnesses that (a) holds. Then (i,j,m) witnesses that every vertex in $N(v) \cap U(i)$ is v-good. Since $|N(v) \cap U(i)| \geq d$, (d) holds. ∎

Lemma 2.7. *Suppose $8dn^{-1} \leq p \leq \frac{1}{24}k^{-r+2}$. If $w \in N(v) \cap Y$ is v-good, then for any $X \subset Y - \{w\}$, with $|X| \leq \gamma$, there exists $S \subset Y - (N(v) \cup X)$ such that $G[S \cup \{w\}] \approx T'$ with root w. Similarly, for all $i \in W$, if $w_1, \ldots, w_k \in U(i)$ are all v-good, then there exists $S \subset Y - N(v)$ such that $G[S \cup \{v, w_1, \ldots, w_k\}] \approx T'$ with root v.*

Proof. Let (i,j,m) witness that w is v-good. Let $B = U(j) \cup U(m)$. We show by induction on $s \in [r-1]$ that there exists $S_s \subset B - (N(v) \cup X)$ such that $T_s = G[S_s \cup \{w\}] \approx T(k,s)$ with root w and all the leaves of T_s are either in $U(j)$ or $U(m)$. First consider the base step $s = 1$. By (i), $|N(w) \cap U(j)| \geq pn/2$. Also $|(N(v) \cap U(j)) \cup X| \leq d + \gamma \leq 2d$. By the hypothesis, $pn/2 \geq k + 2d$. Thus S_1 can be any k-subset of $(N(w) \cap U(j)) - ((N(v) \cap U(j)) \cup X)$. Now consider the inductive step. Without loss of generality, suppose that the leaves of T_{s-1} are contained in $U(m)$. It suffices to show that for any leaf x of T_{s-1}

$$|(N(x) \cap U(j)) - (N(v) \cup X \cup S_{s-1} \cup \{w\} \cup N(S_{s-1} \cup \{w\}))| \geq k.$$

By (i), since $jm \in F$, $|N(x) \cap U(j)| \geq pn/2$. By the choice of j, $|N(x) \cap U(j) \cap N(v)| < d$. By hypothesis $|X| \leq \gamma$. Also, $|S_{s-1} \cup \{w\}| < 2k^{s-1} \leq 2k^{r-2}$. Finally, by (ii), $|N(x) \cap U(j) \cap N(S_{s-1} \cup \{w\})| < 2p^2 n 2k^{s-1} \leq 4p^2 nk^{r-2}$. Thus it suffice to check that $pn/2 \geq 2d + 6p^2 nk^{r-2}$, which is implied by $pn/4 \geq 2d$ and $pn/4 \geq 6p^2 nk^{r-2}$. This in turn follows from the hypothesis $8dn^{-1} \leq p \leq \frac{1}{24}k^{-r+2}$. This completes the proof of the first part. The second part is proved by a similar argument. ∎

Note that the hypothesis of Lemma 2.7 holds for sufficiently large k.

Lemma 2.8. *For sufficiently large k, H is a template for G.*

Proof. Let $v \in N(H)$. For (1), suppose that $X \subset Y$ with $|X| \leq \gamma$. If $(N(v) \cap Y) - X \neq \emptyset$, then, by Lemma 2.6, either every vertex in $N(v) \cap Y$ is v-good or there are at least d vertices in Y that are v-good. In either case there is a v-good vertex $w \in (N(v) \cap Y) - X$. Thus, by Lemma 2.7, there exists $S \subset Y - (N(v) \cup X)$ such that $G[S \cup \{w\}] \approx T'$ with root w. Now consider (2). By Lemma 2.6, there exists a v-good vertex w. By Lemma 2.7, for any $X \subset Y - \{w\}$, with $|X| \leq \gamma$, there exists $S \subset Y - (N(v) \cup X)$ such that $G[S \cup \{w\}] \approx T'$ with root w. For (3), suppose that v is strongly adjacent to H. There exists a vertex $i \in W$ such that $|U(i) \cap N(v)| \geq \alpha/2q = k$. By Lemma 2.6, either every neighbor of v in $U(i)$ is v-good or there exist $j \in W$ such that at least $d(> k)$ neighbors of v in $U(j)$ that are v-good. In either case, there exists $m \in W$ such that $U(m)$ has at least k v-good vertices. Thus by the second part of Lemma 2.7 there exists $S \subset Y$ such that $G[S \cup \{v\}] \approx T'$ with root v. ∎

This completes the proof of Theorem 2.1. ∎

Theorem 2.9. *For all positive integers q and r and sufficiently large k, $\Gamma = \mathrm{Forb}(T(k,r),K_q)$ is not vertex Ramsey.*

Proof. By Theorem 2.1, there exists a graph $H \in \Gamma$ such that H is a template for all graphs $G \in \Gamma$, that induce H. Moreover it follows from condition (iii) that $\omega(H) = q$. Thus for all graphs $G \in \Gamma$ and vertices v in G, H is not an induced subgraph of $G[N(v)]$, i.e., $\lambda_H(G) = 1$. Thus by Theorem 1.1, $\chi_H(G)$ is bounded for all $G \in \Gamma$. It follows that Γ is not vertex Ramsey. ∎

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